Introduction to Markov Decision Processes

Balázs Csanád Csáji

Research Fellow, The University of Melbourne

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Department of Electrical and Electronic Engineering, The University of Melbourne

Overview

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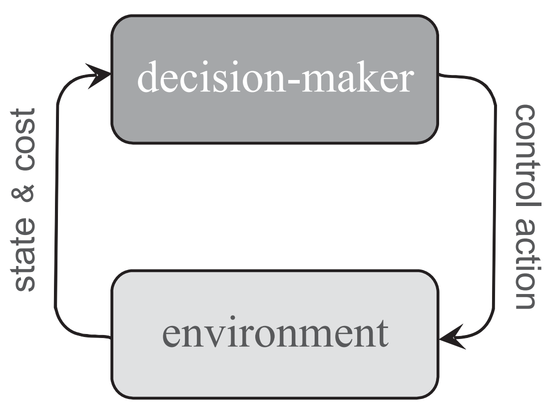
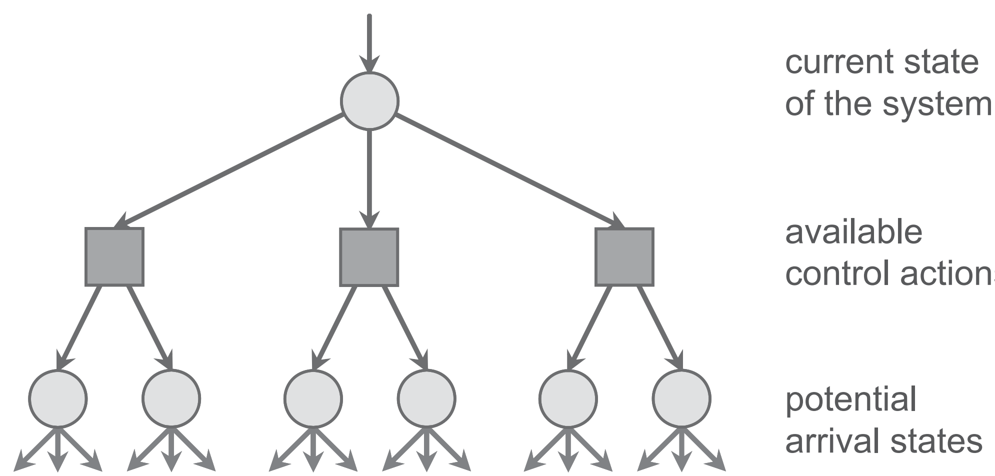
Part I.

Introduction

Motivation: Reinforcement Learning

- Reinforcement learning (RL) is a computational approach to learn from the interaction with an environment based on feedbacks, e.g., rewards.
- An interpretation: consider an agent acting in an uncertain environment and receiving information on the actual states and immediate costs.
- The aim is to learn an efficient behavior (control policy), such that applying this strategy minimizes the expected costs in the long run.





Applications

Some applications of Markov decision processes:

- Optimal Stopping
- Routing
- Maintenance and Repair
- Dispatching & Scheduling
- Inventory Control
- Optimal Control of Queues
- Strategic Asset Pricing
- Dynamic Options
- Insurance Risk Management
- Robot Control
- Logic Games
- Communication Networks
- Dynamic Channel Allocation
- Power Grid Management
- Supply-Chain Management
- Stochastic Resource Allocation
- Sequential Clinical Trials
- PageRank Optimization

Reminder: Markov Chains

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a nondecreasing family of σ-algebras, called a filtration, $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}$.
- Note that $\mathcal{F}_t$ can be interpreted as the information available at time t.
- A stochastic sequence $X = (X_i, \mathcal{F}_i)_{i=0}^{\infty}$ is a Markov chain (w.r.t. $\mathbb{P}$) if

$$\boxed{\mathbb{P}(X_i \in A \,|\, \mathcal{F}_j) = \mathbb{P}(X_i \in A \,|\, X_j),}$$

  for all $0 \leq j \leq i$ and $A \in \mathcal{B}(\mathbb{X})$ ($\mathbb{P}$-a.s.), where $\mathbb{X}$ is the state space of the process and $\mathcal{B}(.)$ denotes a σ-algebra.
- For example, $\mathbb{X} = \mathbb{R}$ and $\mathcal{B}(\mathbb{X})$ denotes the Borel measurable sets.
- For countable state spaces, for example $\mathbb{X} \subseteq \mathbb{Q}^d$, the σ-algebra $\mathcal{B}(\mathbb{X})$ will be assumed to be the set of all subsets of $\mathbb{X}$.

Countable State Spaces

- Henceforth we assume that $\mathbb{X}$ is countable and $\mathcal{B}(\mathbb{X}) = \mathcal{P}(\mathbb{X}) (= 2^{\mathbb{X}})$.
- We say that $\lambda = (\lambda_x : x \in \mathbb{X})$ is a is distribution if $\lambda_x \geq 0$ for all x and

$$\sum_{x \in \mathbb{X}} \lambda_x = 1.$$

- A (potentially infinite) matrix $P = (p_{xy} : x, y \in \mathbb{X})$ is stochastic if its each row $(p_{xy} : y \in \mathbb{X})$ is a distribution.
- A sequence of discrete random variables $(X_i)_{i=0}^{\infty}$ is a (homogeneous) Markov chain with initial distribution λ and transition matrix P if
  - X_0 has distribution λ;
  - For all $i \geq 0$, conditional on $X_i = x$, X_{i+1} has distribution $(p_{xy} : y \in \mathbb{X})$ and is independent of $X_0, \dots, X_{i-1}$.

Transition Probabilities

- We can easily extend the matrix multiplication to the general case:

$$(\lambda P)_y \triangleq \sum_{x \in \mathbb{X}} \lambda_x p_{xy}, \qquad (PQ)_{xz} \triangleq \sum_{y \in \mathbb{X}} p_{xy} q_{yz},$$

  for all (potentially infinite) vector λ and matrices P, Q.

- The (generalized) identity matrix is denoted by I, where $I_{xy} = \delta_{xy}$.

- Matrix powers can be defined as usual, $P^0 \triangleq I$ and $P^{n+1} \triangleq P^n P$.

- If $(X_i)_{i=0}^{\infty}$ is a (discrete, homogeneous) Markov chain then
  - $\mathbb{P}(X_n = x) = (\lambda P^n)_x$,
  - $\mathbb{P}(X_{m+n} = y \,|\, X_m = x) = (P^n)_{xy}$,

  where λ is its initial distribution and P is its transition matrix.

Part II.

The Basic Model of Markov Decision Processes

Markov Decision Processes

A (homogeneous, discrete, observable) Markov decision process (MDP) is a stochastic system characterized by a 5-tuple $\mathcal{M} = \langle \mathbb{X}, \mathbb{A}, \mathcal{A}, p, g \rangle$, where:

- $\mathbb{X}$ is a countable set of discrete states,
- $\mathbb{A}$ is a countable set of control actions,
- $\mathcal{A} : \mathbb{X} \to \mathcal{P}(\mathbb{A})$ is an action constraint function, $\mathcal{A}(x)$ denotes the finite set of allowed actions in state x,
- $p : \mathbb{X} \times \mathbb{A} \to \Delta(\mathbb{X})$ is the transition function, $p(y \,|\, x, a)$ denotes the probability of arriving at state y after taking action $a \in \mathcal{A}(x)$ in a state x,
- $g : \mathbb{X} \times \mathbb{A} \to \mathbb{R}$ is an immediate cost (or reward) function.

Note that $\Delta(\mathbb{X})$ is the set of all probability distributions on $\mathbb{X}$.

Control Policies

The policy defines which action to take depending on the history:

$$\pi_n : \mathbb{X} \times (\mathbb{A} \times \mathbb{X})^n \to \Delta(\mathbb{A}).$$

Unless indicated otherwise, we consider stationary, Markov policies:

- A deterministic (stationary, Markov) policy is $\pi : \mathbb{X} \to \mathbb{A}$.
- A randomized (stationary, Markov) policy is $\pi : \mathbb{X} \to \Delta(\mathbb{A})$.

Each policy induces a (stochastic) transition matrix on the state space:

$$(P_\pi)_{xy} \triangleq \sum_{a \in \mathcal{A}(x)} p(y \,|\, x, a) \pi(x, a).$$

The initial distribution of the states $\lambda \in \Delta(\mathbb{X})$, the transition probabilities p, together with a policy π define a homogeneous Markov chain on $\mathbb{X}$.

Performance Measures: Total Cost

- We aim at finding a policy that minimizes the costs in the long run.
- A common performance measures is the expected discounted cost.
- In this case, the value function of a policy $V^\pi : \mathbb{X} \to \mathbb{R}$ is defined as

$$V^\pi(x) \triangleq \mathbb{E}_\pi \left[\sum_{n=0}^{\infty} \beta^n g(X_n, A_n) \,\middle|\, X_0 = x \right],$$

  where $\beta \in [0, 1)$ is a discount factor and A_n, X_n are random variables with $A_n \sim \pi(X_n)$ and $X_{n+1} \sim p(X_n, A_n)$ ("$\sim$" = "has distribution").
- Function V^π is well-defined (and finite) if, e.g., the immediate-costs are bounded: for all $x \in \mathbb{X}$ and $a \in \mathcal{A}(x)$, we have $|g(x, a)| \leq C$.
- In this latter case, for all state x, we have $|V^\pi(x)| \leq C/(1 - \beta)$.

Performance Measures: Ergodic Cost

- An alternative performance measure is the expected ergodic cost; in this case, the value function of a policy $W^\pi : \mathbb{X} \to \mathbb{R}$ is defined as

$$W^\pi(x) \triangleq \limsup_{k\to\infty} \frac{1}{k} \mathbb{E}_\pi \left[\sum_{n=0}^{k-1} g(X_n, A_n) \;\middle|\; X_0 = x \right].$$

  where, as previously, A_n, X_n are random variables representing the state and action at time n, with $A_n \sim \pi(X_n)$ and $X_{n+1} \sim p(X_n, A_n)$.

- If the state space $\mathbb{X}$ is finite or the costs are bounded, W is finite.

- If $\mathbb{X}$ is infinite, an optimal stationary Markov policy may not exist.

- In many applications the average cost is independent of the initial state, namely, $W^\pi(x)$ is constant (e.g., for finite, irreducible chains).

Optimal Solutions

- In general, we search for a solution that has minimal cost over all polices, with respect to a given performance measure $\mu(x, \pi)$.
- For example, $\mu(x, \pi) = V^{\pi}(x)$ or $\mu(x, \pi) = W^{\pi}(x)$.
- The optimal value function is defined as

$$\mu^*(x) \triangleq \inf_{\pi \in \Pi} \mu(x, \pi),$$

for all state $x \in \mathbb{X}$, where Π denotes the set of all admissible policies.

- For $\varepsilon \geq 0$, a policy is called ε-optimal, if $\mu(x, \pi) \leq \mu^*(x) + \varepsilon$, for all state $x \in \mathbb{X}$. A 0-optimal policy is called optimal.
- Henceforth, we apply the discounted cost criterion, $\mu(x, \pi) = V^{\pi}(x)$.

Optimality Operators

- Let $\mathcal{B}(\mathbb{X}, \mathbb{R})$ denote the set of all bounded real-valued functions on $\mathbb{X}$ with bound $\|g\| / (1 - \beta)$. It contains the value functions of all policies.

- For a function $f \in \mathcal{B}(\mathbb{X}, \mathbb{R})$ we consider two cost operators:

$$(P_a f)(x) \triangleq \mathbb{E}\left[f(X_1) \,|\, X_0 = x, A_0 = a \right],$$

$$(T_a f)(x) \triangleq g(x, a) + \beta (P_a f)(x).$$

- The optimality operators are defined for all state $x \in \mathbb{X}$ by:

$$(P f)(x) \triangleq \inf_{a \in \mathcal{A}(x)} (P_a f)(x),$$

$$(T f)(x) \triangleq \inf_{a \in \mathcal{A}(x)} (T_a f)(x).$$

- Operators T_a and T are often called the Bellman operators.

Optimality Equations

- The Bellman optimality equation for the discounted cost problem is:

$$\boxed{V(x) = (T\,V)(x),}$$

for all state $x \in \mathbb{X}$. Its solution V^* is the optimal value function.

- It can be proven that T is a contraction with Lipschitz constant β:

$$\|T f_1 - T f_2\| \leq \beta \, \|f_1 - f_2\| \, .$$

for all $f_1, f_2 :\in \mathcal{B}(\mathbb{X}, \mathbb{R})$, where $\|\cdot\|$ denotes supremum norm.

- Using the Banach fixed point theorem we get that T has a unique fixed point. Thus, all optimal policies share the same value function.

- Moreover, T is monotone, namely, $f_1 \leq f_2 \Rightarrow T f_1 \leq T f_2$.

Greedy Policies

- For countable state spaces the Bellman operator takes the form:

$$(T\,V)(x) = \min_{a \in \mathcal{A}(x)} \Big[g(x,a) + \beta \sum_{y \in \mathbb{X}} p(y \,|\, x, a) V(y) \Big].$$

- Given a value function V, the greedy policy (w.r.t. V) is defined as

$$\pi(x) \in \arg\min_{a \in \mathcal{A}(x)} \Big[g(x,a) + \beta \sum_{y \in \mathbb{X}} p(y \,|\, x, a) V(y) \Big].$$

- If $V \in \mathcal{B}(\mathbb{X}, \mathbb{R})$ and π is a greedy policy with respect to V, then

$$\| V^* - V^\pi \| \leq \frac{2\beta}{1-\beta} \, \| V - V^* \| \,,$$

- Thus, good approximations to V^* directly lead to efficient policies.

PART III.

Solution Methods

Three Ways

The three classical ways of solving MDPs are:

1. Successive approximations: iteratively computing a value function that approximates V^*. For example, value iteration and Q-learning.

2. Direct policy search: searching for an optimal control policy in the space of policies. For example, policy iteration and policy gradient methods.

3. Linear programming: finding the optimal value function as a solution of a static optimization problem with linear objective function and constraints.

TYPE 1 SOLUTIONS

Successive Approximations

(Value Function Based Methods)

Value Iteration

- Since T is a contraction, the recursive sequence

$$V_{n+1} = T\,V_n$$

converges to V^* for any initial value function $V_0 \in \mathcal{B}(\mathbb{X}, \mathbb{R})$.

- This method is called value iteration. Its main problems are:

1. If $\mathbb{X}$ is very large $\Rightarrow T\,V$ cannot be computed for all states at once.
2. The transition probabilities $p(y\,|\,x, a)$ are often not known in advance.
3. If $\mathbb{X}$ is very large $\Rightarrow V$ cannot be directly stored in the memory.

- A possible solution to problem 1 is asynchronous value iteration: $\widetilde{V}_{n+1}(x) = (T\,\widetilde{V}_n)(x)$ is updated only for $x \in \mathbb{X}_{n+1} \subseteq \mathbb{X}$.
- $\widetilde{V}_n$ converges to V^* if each state is updated infinite many times.

Q-learning

- Simulation based methods can handle unknown transition probabilities.
- The action-value function of a policy $Q^\pi : \mathbb{X} \times \mathbb{A} \to \mathbb{R}$ is

$$Q^\pi(x,a) = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t g(X_t, A_t) \,\middle|\, X_0 = x, A_0 = a\right].$$

- The optimal action-value function, which uniquely exists, is Q^*.
- The one-step version of Watkins' Q-learning rule is as follows.

$$\begin{aligned} Q_{n+1}(x,a) \quad &= \quad (1 - \gamma_n(x,a))Q_n(x,a) + \\ &\qquad \gamma_n(x,a)\left(g(x,a) + \beta \min_{B \in \mathcal{A}(Y)} Q_n(Y,B)\right), \end{aligned}$$

where $\gamma_n \in (0,1)$ is the learning rate, e.g., $\gamma_n(x,a) = 1/n$, and Y is a random variable generated from $(x,a) \in \mathbb{X} \times \mathbb{A}$ by simulation.

Q-learning

- Q-learning is off-policy: it can work independently of the applied policy.
- Assume that the learning rates satisfy for all state x and action a (w.p.1):

$$\sum_{n=0}^{\infty} \gamma_t(x,a) = \infty, \qquad \sum_{n=0}^{\infty} \gamma_t^2(x,a) < \infty.$$

- Under this assumption and if all state-action pairs are continue to update, the sequence Q_t converges to Q^* almost surely for all Q_0.
- Policy π is called soft if $\forall x \in \mathbb{X} : \forall a \in \mathcal{A}(x) : \pi(x,a) > 0$, e.g.,

$$\pi_n(x,a) \triangleq \frac{\exp(-Q_n(x,a)/\tau)}{\sum_{b \in \mathcal{A}(x)} \exp(-Q_n(x,b)/\tau)},$$

where $\tau > 0$ is the Boltzmann temperature. It is a semi-greedy policy.

Stochastic Iterative Algorithms

- Many learning and optimization methods can be written in a general form as a stochastic iterative algorithm. More precisely, for all $z \in \mathcal{Z}$ as

$$\boxed{f_{t+1}(z) = (1 - \gamma_t(z))f_t(z) + \gamma_t(z)((K f_t)(z) + W_t(z)),}$$

  where f_t is a (generalized) value function, operator K acts on value functions, γ_t denotes random stepsizes and W_t is a noise parameter.

- Approximate dynamic programming methods often take the form

$$\Phi(r_{t+1}) = \Pi\Big((1 - \gamma_t)\,\Phi(r_t) + \gamma_t(K_t(\Phi(r_t)) + W_t)\Big),$$

  where $r_t \in \Theta$ is a parameter, Θ is the parameter space, e.g., $\Theta \subseteq \mathbb{R}^d$, $\Phi : \Theta \to \mathcal{F}$ is an approximation function where $\mathcal{F} \subseteq \mathcal{V}$ is a Hilbert space of functions. Function $\Pi : \mathcal{V} \to \mathcal{F}$ is a projection mapping.

TYPE 2 SOLUTIONS

Direct Policy Search

Policy Iteration

- Let π_1 be a policy with value function V^{π_1}. Let π_2 be the greedy policy w.r.t V^{π_1}, then $V^{\pi_2} \leq V^{\pi_1}$ and $V^{\pi_2} = V^{\pi_1} \Rightarrow$ they are both optimal.
- The method of policy iteration works by iteratively evaluating the policy then improving it by the greedy policy with respect to the evaluation.

1. **Initialize** π_0 arbitrarily and **set** the iteration counter, $n := 0$.
2. **Repeat** (iterative evaluation and improvement)
3. **Evaluate** policy π_n by computing its value function V^{π_n}.
4. V^{π_n} is the solution of the linear system $(I - \beta\, P_{\pi_n})\, x = g_{\pi_n}$.
5. **Improve** the policy by setting π_{n+1} to the greedy policy w.r.t. V^{π_n}.
6. **Increase** the iteration counter, $n := n + 1$.
7. **Until** $\pi_n = \pi_{n-1}$ (until no more improvements are possible)

- If $\mathbb{X}$ is finite, it terminates in finite steps with an optimal policy.

Policy Evaluation

- Simulation based methods, e.g., temporal difference learning, can evaluate a policy without knowledge of the transition probabilities.
- The update rule of TD(0) is defined as

$$V_{n+1}(x_n) = V_n(x_n) + \gamma_n(x_n)\Big(g(x_n, a_n) + \beta\, V_n(x_{n+1}) - V_n(x_n)\Big),$$

  where the trajectory $x_0, a_0, x_1, a_1, \ldots$ is generated by Monte Carlo simulation and $\gamma_n(\cdot)$ denote the learning rate.
- Assuming the usual properties about $\gamma_n(\cdot)$ and every state is visited infinitely often, V_n converges to V^π (w.p.1) starting from $V_0 \equiv 0$.
- The optimistic policy iteration with TD(0) converges to V^* (w.p.1).

Policy Gradient

- Assuming that the policy π is parametrized by $\theta \in \mathbb{R}^d$: $a \sim \pi_\theta(x, a)$, the policy gradient method with learning rate γ_n (with $V(\theta) \triangleq V^{\pi_\theta}$)

$$\boxed{\theta_{n+1} = \theta_n - \gamma_n \nabla_\theta V(\theta)|_{\theta=\theta_n} .}$$

- If the gradient estimator is unbiased and $\sum_n \gamma_n = \infty$, $\sum_n \gamma_n^2 < \infty$, the convergence to a local minimum can be guaranteed.

- In order to estimate the gradient, θ can be perturbed $\hat{\theta}_i = \theta_n + \delta\theta_i$.

- In this case, the gradient can be determined by linear regression:

$$\nabla_\theta V(\theta)|_{\theta=\theta_n} \approx (\Delta\Theta^{\mathrm{T}} \Delta\Theta)^{-1} \Delta\Theta^{\mathrm{T}} \Delta J,$$

with $\Delta\Theta \triangleq (\delta\theta_1, \dots, \delta\theta_k)^{\mathrm{T}}$ and $\delta J_i \triangleq J(\hat{\theta}_i) - J(\theta_n)$ rollout returns form $\Delta J \triangleq (\delta J_1, \dots, \delta J_k)^{\mathrm{T}}$, where k is the sample size.

Policy Gradient

- The advantages of policy gradient methods are:
  - They allow incorporation of domain knowledge in the parametrization.
  - Often significantly fewer parameters are enough to represent the optimal policy than the corresponding value function.
  - They are guaranteed to converge while successive approximation methods with function approximation often does not.
  - They can handle continuous state and action spaces, as well.
- The disadvantages of policy gradient methods are:
  - They only converge to a local optima, not to a global one.
  - They are difficult to use in off-policy settings.
  - The convergence rate is often slow in discrete problems.

TYPE 3 SOLUTIONS

Linear Programming

Linear Programming

- We know that the optimal value function satisfies $T\,V^* = V^*$, it is monotone, $V_1 \leq V_2 \Rightarrow T\,V_1 \leq T\,V_2$, and $T^n V \to V^*$ as $n \to \infty$.
- Therefore, V^* is the largest vector satisfying $T\,V \leq V$.
- The optimal value function, V^*, solves the following linear program (LP):

$$\boxed{\begin{array}{ll} \text{maximize} & \displaystyle\sum_{x \in \mathbb{X}} \lambda_x \\ \text{subject to} & \displaystyle\lambda_x \;\leq\; g(x,a) + \beta \sum_{y \in \mathbb{X}} p(y \,|\, x, a)\, \lambda_y \end{array}}$$

  for all state x and action $a \in \mathcal{A}(x)$. If its solution is λ^*, then $\lambda^* = V^*$.

- If $\mathbb{X}$ is finite, it can be solved efficiently by, e.g., interior point methods. In case of infinite $\mathbb{X}$, it can be often approximated by a finite program.

Computational Complexity

- Assume that the MDP is finite, particularly $|\mathbb{X}| = n$ and $|\mathbb{A}| = m$.
- It is equivalent with an LP with n variables and $O(nm)$ constraints.
- Thus, it can be solved in polynomial time, assuming the Turing model.
- Reducing to an MDP is often useful in combinatorial optimization, if we want to show the polynomial computability of a class of problems.
- If the discount factor, β, is fixed, value iteration and policy iteration also runs in polynomial time. However, value iteration is not polynomial in β.
- It is not known whether MDPs can be solved in strongly polynomial time. (Naturally, it is also not known whether LPs can be solved this way.)

Part IV.

Generalizations

Potential Criticisms

- Some possible questions about the presented paradigm:
  1. Isn't the Markov assumption too restrictive?
  2. Can this methodology deal with delayed costs (or rewards)?
  3. What if the immediate costs are not bounded?
  4. Can this theory be extended to system that are not fully observed?
  5. What can we do if the state and action spaces are uncountable?
- Regarding 1: note that Semi-MDPs can be reduced to MDPs.
- Regarding 2: delayed costs do not introduce change to the theory.
- We are going to investigate 3, 4 and 5 now.

Unbounded Costs

- So far we have assumed that the costs are bounded: $\|g\| \leq C$.
- A generalization: it is enough to assume that for all state x there is a number C_x and a constant k such that for all policy π, we have

$$\mathbb{E}_\pi [\, |g(X_{n-1}, A_{n-1})| \mid X_0 = x \,] \leq C_x n^k.$$

- Under the condition above, the value function of policy π satisfy

$$\mathbb{E}_\pi \left[\sum_{n=0}^{\infty} \beta^n g(X_n, A_n) \,\middle|\, X_0 = x \right] \leq C_x \sum_{n=0}^{\infty} \beta^n (n+1)^k < \infty,$$

therefore, the value function V^π remains well-defined (and finite).
- Alternatively, for most of the theory it is enough if we assume that g is bounded (only) from below: $g(x, a) \geq C$ for all state x and action a.

Partial Observability

- A partially observable Markov decision process (POMDP) is an MDP where the states cannot be observed directly.
- For POMDPs the definition of MDPs is extended with an observation set $\mathbb{O}$ and an observation probability function: $q : \mathbb{X} \times \mathbb{A} \to \Delta(\mathbb{O})$.
- $q(z \,|\, x, a)$ is the probability of observing z if we take action a in state x.
- The action constraint function takes the form: $\mathcal{A} : \mathbb{O} \to \mathcal{P}(\mathbb{A})$.
- The policies also depend on the observations instead states, e.g., a randomized Markov policy takes the form of $\pi : \mathbb{O} \to \Delta(\mathbb{A})$.
- In general, policies should depend on the whole history to be optimal.

Belief States

- Belief states are probability distributions over the states, $\mathbb{B} \triangleq \Delta(\mathbb{X})$. They can be interpreted as the agent's ideas about the current state.

- Given a belief state b, an action a and an observation z, the new belief state $\tau(b, a, z)$ can be computed by Bayes rule:

$$\tau(b, a, z)(y) = \frac{\sum_{x \in \mathbb{X}} p(z, y \,|\, x, a) b(x)}{p(z \,|\, b, a)},$$

  where $p(z, y \,|\, x, a) \triangleq q(z \,|\, y, a) p(y \,|\, x, a)$ and

$$p(z \,|\, b, a) \triangleq \sum_{x, y \in \mathbb{X}} p(z, y \,|\, x, a) b(x).$$

- They are sufficient statistics: an optimal policy can found based on them.

Belief State MDPs

- A POMDP can be transformed into a fully observable MDP.
- The new MDP is called the belief state MDP. Its state space is $\mathbb{B}$, the action space remains $\mathbb{A}$ and the transition probabilities are:

$$p(b_2 \mid b_1, a) = \begin{cases} p(z \mid b_1, a) & \text{if } \; b_2 = \tau(b_1, a, z) \; \text{ for some } \; z \\ 0 & \text{otherwise} \end{cases}$$

The new immediate cost function for all $b \in \mathbb{B}$, $a \in \mathbb{A}$ is given by

$$g(b, a) = \sum_{x \in \mathbb{X}} b(x)\, g(x, a),$$

consequently, the optimal value function of the belief state MDP is

$$\widetilde{V}^*(b) = \min_{a \in \mathcal{A}(b)} \Big[g(b, a) + \beta \sum_{z \in \mathbb{O}} p(z \mid b, a)\, \widetilde{V}^*(\tau(b, a, z)) \Big].$$

General Measurable Spaces

In case the state $\mathbb{X}$ or the action space $\mathbb{A}$ is non-discrete, we need:

- The state space $\mathbb{X}$ is measurable space endowed with a σ-field $\mathcal{X}$.
- The action space $\mathbb{A}$ is measurable space endowed with a σ-field $\mathcal{A}$.
- For all state $x \in \mathbb{X}$, the set of allowed actions $\mathcal{A}(x)$ is measurable.
- The cost function g is a measurable function on $(\mathbb{X} \times \mathbb{A}, \mathcal{X} \times \mathcal{A})$.
- $p(\cdot \mid \cdot)$ is a transition probability from $(\mathbb{X} \times \mathbb{A}, \mathcal{X} \times \mathcal{A})$ to $(\mathbb{X}, \mathcal{X})$:
  - For all x, a: $p(\cdot \mid x, a)$ is a probability measure on $(\mathbb{X}, \mathcal{X})$ and,
  - For all E: $p(E \mid \cdot)$ is a measurable function on $(\mathbb{X} \times \mathbb{A}, \mathcal{X} \times \mathcal{A})$.

For discrete MDPs, these assumptions always hold, since
in that case $\mathcal{X} \triangleq \mathcal{P}(\mathbb{X})(= 2^{\mathbb{X}})$ and $\mathcal{A} \triangleq \mathcal{P}(\mathbb{A})(= 2^{\mathbb{A}})$.

PART V.

Summary and Literature

Summary

- Markov decision processes are controlled Markov chains together with an immediate-cost function and an optimization criterion.
- They have a large number of practical and theoretical applications.
- The basic concepts are: policy, value function and optimality equations.
- The three classical types of solution methods are:
  1. Successive Approximations (e.g., value iteration, Q-learning)
  2. Direct Policy Search (e.g., policy iteration, policy gradient)
  3. Linear Programming (e.g., interior point, ellipsoid or simplex methods)
- The theory can handle semi-Markov and partially observable problems and can be generalized to general measurable state and action spaces.

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Thank you for your attention!

bcsaji@unimelb.edu.au