MTA SZTAKI
Hungarian Academy of Sciences
Institute for Computer Science and Control

# Closed-Loop Applicability of the Sign-Perturbed Sums Method

Balázs Csanád Csáji[1] Erik Weyer[2]

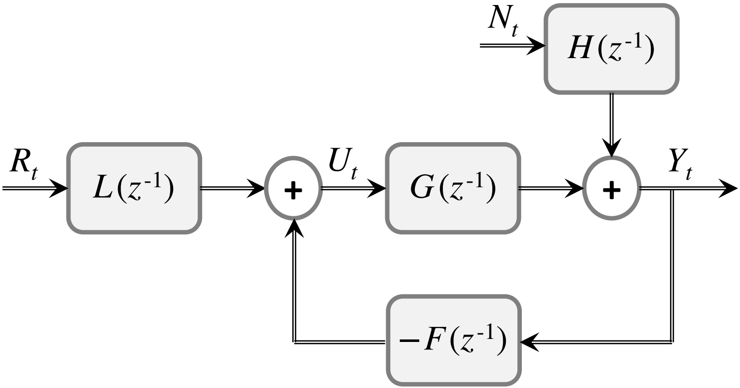
[1]Institute for Computer Science and Control (SZTAKI), Hungarian Academy of Sciences, Hungary

[2]Department of Electrical and Electronic Engineering (EEE), University of Melbourne, Australia

54th IEEE CDC, Osaka, Japan, December 15-18, 2015

# Overview

# Closed-Loop General Linear System




$t$ : (discrete) time, $Y_t$ : output, $U_t$ : input, $N_t$ : noise, $R_t$ : reference,

$F, G, H, L$ (causal) rational transfer functions, $z^{-1}$ : backward shift.

# Closed-Loop General Linear System

**Dynamical System: General Linear**

$$Y_t \triangleq G(z^{-1};\theta^*)\,U_t + H(z^{-1};\theta^*)\,N_t$$

$t$ : (discrete) time, $Y_t$ : output, $U_t$ : input, $N_t$ : noise, $R_t$ : reference, $G, H$ : transfer functions, $z^{-1}$ : backward shift, $\theta^*$ : true parameter.

**Controller: Closed-Loop with Reference Signal**

$$U_t \triangleq L(z^{-1};\eta^*)\,R_t - F(z^{-1};\eta^*)\,Y_t$$

$L, F$ : transfer functions parametrized independently of $G, H$.

# Main Assumptions

(A1) The "true" systems generating $\{Y_t\}$ and $\{U_t\}$ are in the model classes; $G$ and $H$ have known orders.

(A2) Transfer function $H(z^{-1};\theta)$ has a stable inverse, and $G(0;\theta) = 0$ and $H(0;\theta) = 1$, for all $\theta \in \Theta$.

(A3) The noise sequence $\{N_t\}$ is independent, and each $N_t$ has a symmetric probability distribution about zero.

(A4) The initialization is known, $Y_t = N_t = R_t = 0$, $t \leq 0$.

(A5) The subsystems from $\{N_t\}$ and $\{R_t\}$ to $\{Y_t\}$ are asymptotically stable and have no unstable hidden modes.

(A6) Reference signal $\{R_t\}$ is independent of the noise $\{N_t\}$.

# Review: SPS for Open-Loop Systems

## General Linear Systems

$$Y_t \triangleq G(z^{-1};\theta^*)\,U_t + H(z^{-1};\theta^*)\,N_t$$

- Sign-Perturbed Sums (SPS) is a finite sample system identification method which can build confidence regions.
- SPS is distribution-free, it can work for any symmetric noise.
- The confidence set has exact confidence probability (user-chosen).
- The SPS sets are build around the prediction error estimate.
- SPS is strongly consistent (for lin. reg.).
- The sets of SPS are star convex (for lin. reg.).
- Efficient ellipsoidal outer approximations exists (for lin. reg.).

# Open-Loop Prediction Error Estimate

## Prediction Error or Residual (for parameter $\theta$)

$$\widehat{\varepsilon}_t(\theta) \triangleq H^{-1}(z^{-1};\theta)\left(Y_t - G(z^{-1};\theta)\, U_t\right)$$

Note that $\widehat{\varepsilon}_t(\theta^*) = N_t$, hence, it is accurate for $\theta = \theta^*$.

## Prediction Error Estimate (for model class $\Theta$)

$$\hat{\theta}_{\mathrm{PEM}} \triangleq \operatorname*{arg\,min}_{\theta\in\Theta} \mathcal{V}(\theta \mid \mathcal{Z}) = \operatorname*{arg\,min}_{\theta\in\Theta} \sum_{t=1}^{n} \widehat{\varepsilon}_t^{\,2}(\theta)$$

where $\mathcal{Z}$ is the available data: finite realizations of $\{Y_t\}$ and $\{U_t\}$.

In general, there is no closed-form solution for PEM.

# Open-Loop Prediction Error Equation

The PEM estimate can be found, e.g., by using the equation

**PEM Equation**

$$\nabla_\theta \mathcal{V}(\hat{\theta}_{\mathrm{PEM}} \,|\, \mathcal{Z}) \,=\, \sum_{t=1}^{n} \psi_t(\hat{\theta}_{\mathrm{PEM}})\,\widehat{\varepsilon}_t(\hat{\theta}_{\mathrm{PEM}}) \,=\, 0$$

where $\psi_t(\theta)$ is the negative gradient of the prediction error,

$$\psi_t(\theta) \triangleq -\nabla_\theta\, \widehat{\varepsilon}_t(\theta).$$

These gradients can be directly calculated in terms of the defining polynomials of the rational transfer functions $G$ and $H$.

# Perturbed Samples: Open-Loop Case

## Perturbed Output Trajectories

$$\bar{Y}_t(\theta, \alpha_i) \triangleq G(z^{-1};\theta)\, U_t + H(z^{-1};\theta)\, (\alpha_{i,t}\, \widehat{\varepsilon}_t(\theta))$$

where $\{\alpha_{i,t}\}$ are random signs: $\alpha_{i,t} = \pm 1$ with probability $\frac{1}{2}$ each.

Recall that $\psi_t(\theta)$ is a linear filtered version of $\{Y_t\}$ and $\{U_t\}$,

$$\psi_t(\theta) = W_0(z^{-1};\theta)\, Y_t + W_1(z^{-1};\theta)\, U_t,$$

where $W_0$ and $W_1$ are vector-valued, and $\psi_t(\theta) \in \mathbb{R}^d$.

## Perturbed (Negative) Gradients

$$\bar{\psi}_t(\theta, \alpha_i) \triangleq W_0(z^{-1};\theta)\, \bar{Y}_t(\theta, \alpha_i) + W_1(z^{-1};\theta)\, U_t$$

# Sign-Perturbed Sums: Open-Loop Case

## Reference and $m-1$ Sign-Perturbed Sums

$$S_0(\theta) \triangleq \Psi_n^{-\frac{1}{2}}(\theta) \sum_{t=1}^{n} \psi_t(\theta)\, \widehat{\varepsilon}_t(\theta)$$

$$S_i(\theta) \triangleq \bar{\Psi}_n^{-\frac{1}{2}}(\theta,\alpha_i) \sum_{t=1}^{n} \bar{\psi}_t(\theta,\alpha_i)\, \alpha_{i,t}\, \widehat{\varepsilon}_t(\theta)$$

where $\Psi_n$ and $\bar{\Psi}_n$ are (sign-perturbed) covariances estimates

$$\Psi_n(\theta) \triangleq \frac{1}{n} \sum_{t=1}^{n} \psi_t(\theta)\psi_t^{\mathrm{T}}(\theta)$$

$$\bar{\Psi}_n(\theta,\alpha_i) \triangleq \frac{1}{n} \sum_{t=1}^{n} \bar{\psi}_t(\theta,\alpha_i)\bar{\psi}_t^{\mathrm{T}}(\theta,\alpha_i)$$

# Non-Asymptotic Confidence Regions: Open-Loop Case

$\mathcal{R}(\theta)$ is the rank of $\|S_0(\theta)\|^2$ among $\{\|S_i(\theta)\|^2\}$ (with tie-breaking).

**SPS Confidence Regions for General Linear Systems**

$$\widehat{\Theta}_n \triangleq \left\{ \theta \in \mathbb{R}^d \,:\, \mathcal{R}(\,\theta\,) \leq m - q \right\}$$

where $m > q > 0$ are user-chosen (integer) parameters.

We have $S_0(\hat{\theta}_{\mathrm{PEM}}) = 0$, thus, $\hat{\theta}_{\mathrm{PEM}} \in \widehat{\Theta}_n$, if it is non-empty.

**Exact Confidence of SPS for General Linear Systems**

$$\mathbb{P}\big(\,\theta^* \in \widehat{\Theta}_n\,\big) \;=\; 1 - \frac{q}{m}$$

# Closed-Loop Prediction Error Methods (PEMs)

- Direct Identification
  (Simply neglect the controller, treat the system as the inputs were independent, i.e., if the system operated in open-loop).
- Indirect Identification
  (If the controller is known, treat the reference signal as the input, leading to a reformulated open-loop system).
- Joint Input-Output Identification
  (Identify both the system and the controller as if the observations would come from a system with vector-valued outputs).

# Direct Identification

## Direct Identification (PEM)

- Goal: to estimate $\theta^*$, i.e., to identify $H$ and $G$.
- Assumption: controller is informative.
- Idea: feedback is neglected.
- Method: SISO Open-Loop PEM (original system).

Simply neglecting the feedback does not work for SPS, as

$$\{Y_t\} \quad \text{and} \quad \{\bar{Y}_t(\theta^*, \alpha_1)\}, \dots, \{\bar{Y}_t(\theta^*, \alpha_{m-1})\}$$

does not have the same distribution (essential for exact confidence).

The alternative outputs should be built using alternative inputs.

# Closed-Loop SPS for Direct PEM

Assume that the controller can be simulated (black box).

Then, the alternative output trajectories can be redefined as

**Direct SPS: Perturbed Output Trajectories**

$$\widetilde{Y}_t(\theta, \alpha_i) \triangleq G(z^{-1}; \theta)\, \bar{U}_t(\theta, \alpha_i) + H(z^{-1}; \theta)\, (\alpha_{i,t}\, \widehat{\varepsilon}_t(\theta))$$

using alternative feedbacks given the alternative outputs

**Direct SPS: Alternative Feedbacks**

$$\bar{U}_t(\theta, \alpha_i) \triangleq L(z^{-1}; \eta^*)\, R_t - F(z^{-1}; \eta^*)\, \widetilde{Y}_t(\theta, \alpha_i)$$

The exact confidence probability of Direct SPS is then guaranteed.

# Indirect Identification

## Indirect Identification (PEM)

- Goal: to estimate $\theta^*$, i.e., to identify $H$ and $G$.
- Assumptions: controller is known, $\{R_t\}$ is measurable.
- Idea: restate as an open-loop system, treat $\{R_t\}$ as inputs.
- Method: SISO Open-Loop PEM (reformulated system).

An alternative open-loop system can be formulated as

$$Y_t \;=\; G_0(z^{-1};\kappa^*)\, R_t + H_0(z^{-1};\kappa^*)\, N_t$$

where the parametrization, $\kappa$, can be different and

$$\begin{aligned} G_0(z^{-1};\kappa^*) \;&\triangleq\; (1+GF)^{-1}GL \\ H_0(z^{-1};\kappa^*) \;&\triangleq\; (1+GF)^{-1}H \end{aligned}$$

# Closed-Loop SPS for Indirect PEM

Then, open-loop SPS can applied by treating $\{R_t\}$ as the input.

In order to test $\theta$, the alternative $\kappa$ should be first computed from

$$\begin{aligned}(1+G(\theta)F)^{-1}G(\theta)L &= G_0(\kappa)\\ (1+G(\theta)F)^{-1}H(\theta) &= H_0(\kappa)\end{aligned}$$

If an (exact or approximate) solution is given by $\kappa = g(\theta)$, then

## Indirect SPS Confidence Regions

$$\widehat{\Theta}_n^{id} \triangleq \{\, \theta \in \Theta \,:\, \mathcal{R}(g(\theta)) \leq m - q \,\}$$

which results in exact confidence under the additional assumption

(A7) Parameter transformation $g$ satisfies $g(\theta^*) = \kappa^*$.

# Joint Input-Output Identification

## Joint Input-Output Identification (PEM)

- Goal: to estimate $(\theta^*, \eta^*)$, the controller is also identified.
- Assumption: no reference signal (for simplicity).
- Idea: reformulate as an autonomous vector-valued system.
- Method: MIMO Open-Loop PEM (vector-valued system).

$[Y_t, U_t]^{\mathrm{T}}$ is treated as output of a vector-valued autonomous system

$$Z_t \triangleq \begin{bmatrix} Y_t \\ U_t \end{bmatrix} = \begin{bmatrix} (I + GF)^{-1} H \\ -F(I + GF)^{-1} H \end{bmatrix} N_t = \widetilde{H}(z^{-1}, \kappa^*)\, N_t,$$

driven by symmetric and independent noise terms $\{N_t\}$.

Thus, a vector-valued variant of SPS is needed (future research).

# Experimental Results

**Closed-Loop ARX with Reference Signal:**

$$\begin{aligned} y_t &\triangleq a^* y_{t-1} + b^* u_{t-1} + n_t \\ u_t &\triangleq r_t - c^* y_t \end{aligned}$$

with reference $r_t \triangleq d^* r_{t-1} + w_t$, where $\{w_t\}$ are i.i.d., $U(0,1)$.

For indirect identification the system can be rewritten as

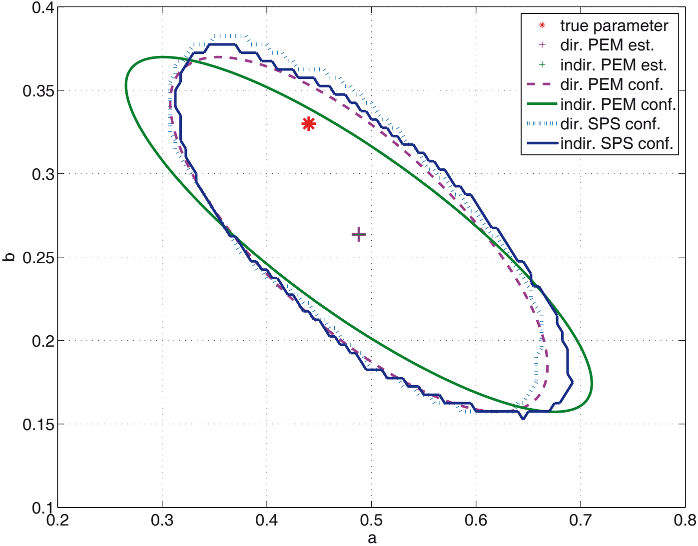
$$y_t = (a^* - b^* c^*) y_{t-1} + b^* r_{t-1} + n_t$$

based on which the indirect SPS confidence set is

$$\widehat{\Theta}_n^{id} = \left\{ (a,b)^{\mathrm{T}} \in \mathbb{R}^2 : \mathcal{R}((a - bc^*, b)^{\mathrm{T}}) \leq m - q \right\}$$

assuming a known controller, i.e., constant $c^*$ is available.

# Experimental Results: Closed-Loop ARX with Reference



# Summary and Conclusion

- Sign-Perturbed Sums (SPS) is a non-asymptotic system identification method which can build exact confidence regions for general linear systems under mild statistical assumptions.
- Originally, SPS was introduced for open-loop systems, where the confidence set is built around the prediction error estimate.
- Here, we showed that the favorable properties of SPS mentioned above can be carried over to closed-loop systems.
- The direct-, the indirect-, and the joint input-output closed-loop approaches of the prediction error method were addressed.
- Closed-loop variants of SPS were discussed for the direct and the indirect cases, both leading to exact confidence regions.
- The joint input-output approach was also mentioned, but left for future research: it requires a vector-valued extension of SPS.

# Thank you for your attention!

✉ balazs.csaji@sztaki.mta.hu